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Nonlinear Dynamic Maximum Power Theorem, with Numerical Method

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ABSTRACT

This paper considers the problem of maximizing the energy or average power transfer from a nonlinear dynamic n-port source. The main theorem includes as special cases the standard linear result $\underset{\sim}{Y}_{load} = \underset{\sim}{Y}^*_{source}$ and a recent finding for nonlinear resistive networks. An operator equation for the optimal output voltage $\hat{v}(\cdot)$ is derived, and a numerical method for solving it is given.

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I. Introduction

This paper addresses the problem of extracting the maximum energy or average power from a source with the topology shown in Fig. 1. As in[1] [1], the problem is formulated as finding the optimal output voltage $\hat{\underset{\sim}{v}}(\cdot)$ for each current source waveform $\underset{\sim}{i}_s(\cdot)$ rather than finding a load that maximizes the power.

The central result is the operator equation (6) for $\hat{\underset{\sim}{v}}(\cdot)$. Theorem 1 gives conditions that guarantee uniqueness and global optimality of the solution: the standard result for linear systems [1] and recent work on resistive nonlinear systems [2] follow as special cases. Equation (11) defines a practical algorithm for solving (6), and Theorem 2 gives conditions that guarantee convergence.

The solution $\hat{\underset{\sim}{v}}(\cdot)$ can be of engineering value in two ways. First, the average power $\overline{P}(\hat{\underset{\sim}{v}})$ tells us the optimal performance that is possible in principle. Second, $\hat{\underset{\sim}{v}}(\cdot)$ itself is a concrete design goal. If the source admittance operator $\underset{\sim}{F}$ is continuous, a load for which the output approximates $\hat{\underset{\sim}{v}}(\cdot)$ (in the Hilbert space norm used in this work) will absorb an average power that approximates $\overline{P}(\hat{\underset{\sim}{v}})$.

1. Reference [1] actually deals with the dual network, where the source appears in Thevenin form.

II. Results

2.1) Notation and Definitions

Let L be any real inner product space and $\hat{L}$ any linear subspace of L. An operator F: $\hat{L} \to L$ is said to be

a) strictly increasing if

$$\langle F(y) - F(x), y-x \rangle > 0, \forall x \neq y \in \hat{L}, \quad (1)$$

b) uniformly increasing if for some $\delta > 0$,

$$\langle F(y) - F(x), y-x \rangle \geq \delta \| y-x \|^2, \forall x,y \in \hat{L}, \quad (2)$$

c) Lipschitz continuous if for some $K \geq 0$,

$$\| F(y) - F(x) \| \leq K \| y-x \|, \forall x,y \in \hat{L}. \quad (3)$$

Let L, L' be any real inner product spaces and $\mathcal{L}(L,L')$ denote the space of continuous linear maps from L to L', with the operator norm [3, p.53]. For $A \in \mathcal{L}(L,L')$, let A^{adj} denote the adjoint of A.

Given an operator F: L → L' and $x,h \in L$, suppose there exists an element denoted $\delta F(x,h)$ of L' such that

$$\lim_{t \to 0^+} \left\| \frac{F(x+th) - F(x)}{t} - \delta F(x,h) \right\|_{L'} = 0.$$

Then $\delta F(x,h)$ is called the Gâteaux variation of F at x for the increment h [4,p.251]. If $\delta F(x,h)$ exists for all $x,h \in L$, and if for each $x \in L$ the map $h \to \delta F(x,h)$ is an element of $\mathcal{L}(L,L')$, then F is said to be Gâteaux differentiable on L. In this case the map $x \to \delta F(x,\cdot)$ is called the Gâteaux derivative of F and denoted DF: $L \to \mathcal{L}(L,L')$ [4,pp.255-256]. Similarly

$\delta F(x,\cdot)$ is denoted $DF(x)\varepsilon L(L,L')$, and $\delta F(x,h)$ is denoted $(DF(x))h\varepsilon L'$. The value of using the Gâteaux derivative rather than the more restrictive Fréchet derivative [4,Chap.3] will become apparent in section 3.1.

The Hilbert space L_n^2 is the set of all measurable functions $\underset{\sim}{x}: \mathbb{R} \to \mathbb{R}^n$ such that the integral of $x_j^2(\cdot)$ over $\mathbb{R}$ is finite, $j=1,\ldots,n$, equipped with the usual inner product $\langle\cdot,\cdot\rangle$ and norm, $\|\underset{\sim}{x}\| \triangleq \langle x,x\rangle^{1/2}$.

For each $T > 0$, $L_{n,T}^2$ is the set of all periodic measurable functions $\underset{\sim}{x}: \mathbb{R} \to \mathbb{R}^n$ with period T such that the integral over one period of $x_j^2(\cdot)$ is finite, $j=1,\ldots,n$. It is a Hilbert space with the "average power" inner product

$$\langle \underset{\sim}{x},\underset{\sim}{y}\rangle_T \triangleq \frac{1}{T}\int_0^T \underset{\sim}{x}(t)\cdot\underset{\sim}{y}(t)\, dt, \tag{4}$$

where $\underset{\sim}{x}\cdot\underset{\sim}{y}$ is the Euclidean inner product on $\mathbb{R}^n$. The norm on $L_{n,T}^2$ is denoted $\|\underset{\sim}{x}\|_T \triangleq \langle\underset{\sim}{x},\underset{\sim}{x}\rangle_T^{1/2}$.

2.2) <u>Main Theorem</u>

<u>Theorem 1</u> (Maximum Average Power in the Periodic Steady State)

Fix $T > 0$ and let the n-port N_1 in Fig. 1 be characterized by an admittance operator $\underset{\sim}{F}: L_T \to L_T$, where[2] L_T is any linear subspace of $L_{n,T}^2$. Suppose $\underset{\sim}{F}$ is Gâteaux differentiable on L_T and that the associated operator $\underset{\sim}{H}: L_T \to L_T$, given by

$$\underset{\sim}{H}: \underset{\sim}{v} \mapsto \underset{\sim}{F}(\underset{\sim}{v}) + (D\underset{\sim}{F}_{(\underset{\sim}{v})})^{adj}\,\underset{\sim}{v}\ , \tag{5}$$

2. Thus, if $\underset{\sim}{v}(\cdot)$ has period T and lies in $L_{n,T}^2$, the response $\underset{\sim}{i}(\cdot)$ of N_1 cannot have subharmonics.

is strictly increasing.

Then for each $\underset{\sim}{i}_s \varepsilon \underset{\sim}{H}(L_T)$ there is a unique solution $\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s) \varepsilon L_T$ to

$$\underset{\sim}{i}_s = \underset{\sim}{H}(\underset{\sim}{v}), \tag{6}$$

and the average power[3] absorbed by the load,

$$\overline{P}(\underset{\sim}{v}) \triangleq \langle \underset{\sim}{i}_o, \underset{\sim}{v} \rangle_T = \langle \underset{\sim}{i}_s - \underset{\sim}{F}(\underset{\sim}{v}), \underset{\sim}{v} \rangle_T , \tag{7}$$

has a unique global maximum over L_T, which is attained at $\underset{\sim}{v} = \hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)$.

<u>Corollary</u> (Maximum Total Energy for Transients)

Let L be a linear subspace of L_n^2, and substitute L for L_T in the assumptions of Theorem 1. Then the same conclusions[4] hold, but with $\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s) \varepsilon L$ maximizing the total energy $E(\underset{\sim}{v}) \triangleq \langle \underset{\sim}{i}_s - \underset{\sim}{F}(\underset{\sim}{v}), \underset{\sim}{v} \rangle$ over L.

Note that in general $\underset{\sim}{F}$ can be nonlinear and time-varying.

In applications one might wish to restrict attention to currents and voltages in $L^2_{n,T}$ with additional properties such as continuity or boundedness. This is the reason for introducing $L_T \subset L^2_{n,T}$ in the formulation of Theorem 1.

The essential idea behind the theorem is that a solution $\hat{\underset{\sim}{v}}(\cdot)$ of (6) is a stationary point of $\overline{P}$: $L_T \to \mathbb{R}$, and the monotonicity assumption on $\underset{\sim}{H}$ guarantees that $\overline{P}$ is strictly concave. Details follow.

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3. A more explicit, but cumbersome, notation would be $\overline{P}(\underset{\sim}{v}, \underset{\sim}{i}_s)$. Using it, Theorem 1 states that $\forall \underset{\sim}{v}, \underset{\sim}{i}_s \varepsilon L_T$, $\overline{P}(\underset{\sim}{v}, \underset{\sim}{i}_s) < \overline{P}(\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s), \underset{\sim}{i}_s)$ if $\underset{\sim}{v} \neq \hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)$.

4. For the Corollary, the adjoint is of course taken with respect to the inner product on L_n^2 rather than $\langle \, , \rangle_T$.

Proof of Theorem 1

Uniqueness of the solution to (6) follows from the fact that $\underset{\sim}{H}$ is strictly increasing. By the chain rule for the composition of Fréchet- and Gâteaux-differentiable functions [4,p.253] $\overline{P}$ is Gâteaux differentiable and for all $\underset{\sim}{x},\underset{\sim}{h}\varepsilon L_T$,

$$(D\overline{P}(\underset{\sim}{x}))\underset{\sim}{h} = \langle \underset{\sim}{i}_s - \underset{\sim}{F}(\underset{\sim}{x}),\underset{\sim}{h}\rangle_T - \langle (D\underset{\sim}{F}(\underset{\sim}{x}))\underset{\sim}{h},\underset{\sim}{x}\rangle_T =$$

$$\langle \underset{\sim}{i}_s - \underset{\sim}{F}(\underset{\sim}{x}) - (D\underset{\sim}{F}(\underset{\sim}{x}))^{adj}\ \underset{\sim}{x},\underset{\sim}{h}\rangle_T =$$

$$\langle \underset{\sim}{i}_s - \underset{\sim}{H}(\underset{\sim}{x}),\underset{\sim}{h}\rangle_T. \tag{8}$$

Thus if $\underset{\sim}{i}_s\varepsilon \underset{\sim}{H}(L_T)$,

a) $D\overline{P}(\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)) = \underset{\sim}{0}\,\varepsilon\, L(L_T,L_T)$,

b) given any $\underset{\sim}{x},\underset{\sim}{y}\ \varepsilon L_T$, the map $\lambda \to \overline{P}[\underset{\sim}{x}+\lambda(\underset{\sim}{y}-\underset{\sim}{x})]$ is differentiable at each $\lambda\varepsilon\mathbb{R}$, and

c) $\frac{d}{d\lambda}\,\overline{P}[\underset{\sim}{x}+\lambda(\underset{\sim}{y}-\underset{\sim}{x})] = \langle \underset{\sim}{i}_s - \underset{\sim}{H}[\underset{\sim}{x}+\lambda(\underset{\sim}{y}-\underset{\sim}{x})],\ (\underset{\sim}{y}-\underset{\sim}{x})\rangle_T$.

To show that $\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)$ globally optimizes $\overline{P}$, fix $\underset{\sim}{i}_s\varepsilon \underset{\sim}{H}(L_T)$, let $\hat{\underset{\sim}{v}} = \hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)$, and choose any $\underset{\sim}{v}\varepsilon L_T$, $\underset{\sim}{v} \neq \hat{\underset{\sim}{v}}$. Then

$$\overline{P}(\underset{\sim}{v}) - \overline{P}(\hat{\underset{\sim}{v}}) =$$

$$\overline{P}[\hat{\underset{\sim}{v}}+\lambda(\underset{\sim}{v}-\hat{\underset{\sim}{v}})]\Big|_{\lambda=1} - \overline{P}[\hat{\underset{\sim}{v}}+\lambda(\underset{\sim}{v}-\hat{\underset{\sim}{v}})]\Big|_{\lambda=0} =$$

$$\int_0^1 \{\frac{d}{d\lambda}\ \overline{P}[\hat{\underset{\sim}{v}}+\lambda(\underset{\sim}{v}-\hat{\underset{\sim}{v}})]\}\ d\lambda. \tag{9}$$

Using c), the integrand above is

$$\left\langle \underset{\sim}{i}_s - \underset{\sim}{H}[\hat{\underset{\sim}{v}}+\lambda(\underset{\sim}{v}-\hat{\underset{\sim}{v}})],\ \underset{\sim}{v}-\hat{\underset{\sim}{v}} \right\rangle_T =$$

(since $\underset{\sim}{i}_s = \underset{\sim}{H}(\hat{\underset{\sim}{v}})$)

$$\frac{-1}{\lambda}\left\langle \underset{\sim}{H}[\hat{\underset{\sim}{v}}+\lambda(\underset{\sim}{v}-\hat{\underset{\sim}{v}})] - \underset{\sim}{H}(\hat{\underset{\sim}{v}}),\ [\hat{\underset{\sim}{v}}+\lambda(\underset{\sim}{v}-\hat{\underset{\sim}{v}})] - [\hat{\underset{\sim}{v}}] \right\rangle_T,\ \forall\lambda>0,$$

and the integrand vanishes at $\lambda=0$. The inner product above is strictly positive for $\lambda\neq 0$ since $\underset{\sim}{H}$ is strictly increasing by assumption. Thus the integrand in (9) is negative for $\lambda>0$ and zero for $\lambda=0$, so $\overline{P}(\underset{\sim}{v}) < \overline{P}(\hat{\underset{\sim}{v}})$ as claimed. ■

The proof of the Corollary is essentially identical and will be omitted.

2.3) Relation to "Impedance Matching" Ideas

The emphasis in this paper is on finding the optimal output voltage $\hat{\underset{\sim}{v}}(\cdot)$, not the optimal load. But the relation to impedance matching ideas deserves comment.

If the load in Fig. 1 is taken to be the (generally noncausal) admittance $\underset{\sim}{G}_{opt}: L_T \to L_T$, defined by

$$\underset{\sim}{G}_{opt}: \underset{\sim}{v} \mapsto (D\underset{\sim}{F}(\underset{\sim}{v}))^{adj}\,\underset{\sim}{v}\ , \qquad (10)$$

then the network is uniquely solvable given any $\underset{\sim}{i}_s \in H(L_T)$, and the output voltage $\underset{\sim}{v}(\cdot)$, which necessarily equals $\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)$, globally optimizes $\overline{P}$. This generally noncausal load is "matched" to the source for all inputs $\underset{\sim}{i}_s \in \underset{\sim}{H}(L_T)$, and this result holds generally for a nonlinear, time-varying, even noncausal source admittance $\underset{\sim}{F}$. The reader can easily verify that in the LTI case (10) reduces to the standard linear theorem $\underset{\sim}{Y}_{load}(j\omega) = \underset{\sim}{Y}^*_{source}(j\omega)$. More detail for the linear 1-port case is given in Section 3.1.

Of course in practice one has a causal load, usually predetermined, and wishes to couple it to the source through a lossless matching network designed to maximize the absorbed power over a range of inputs. In the linear case this important problem is called "broadband matching" [5-8]. We note that in both the linear and nonlinear cases the problem can be viewed as compensating or coupling to a predetermined load using lossless elements in such a way that the response approximates that of the noncausal exact match $\underset{\sim}{G}_{opt}$ over the input range of interest.

For a particular drive $\underset{\sim}{i}_s$, the situation is somewhat different. The optimal voltage $\hat{\underset{\sim}{v}}(\cdot)$ is unique, but the optimal load is not: the only requirement on $\underset{\sim}{G}$ is that $\underset{\sim}{G}(\hat{\underset{\sim}{v}}) = \underset{\sim}{G}_{opt}(\hat{\underset{\sim}{v}})$. In the linear case where $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are respectively represented by admittance matrices $\underset{\sim}{Y}_S(j\omega)$ and $\underset{\sim}{Y}_L(j\omega)$, there are in general infinitely many optimal, positive semidefinite choices of $\underset{\sim}{Y}_L$ at a given ω for which the network is uniquely solvable [9]. The problem of finding solutions in particular classes, such as the class of resistive loads, is studied in [10].

2.4) Numerical Algorithm

Equation (8) shows that $\underset{\sim}{i}_s - \underset{\sim}{H}(\underset{\sim}{v})$ is the gradient [3,p.72], [4,p.196] of $\overline{P}$ at $\underset{\sim}{v}$, $\forall\ \underset{\sim}{i}_s, \underset{\sim}{v} \in L_T$. This suggests that we attempt to maximize $\overline{P}$ by a simple "hill-climbing" algorithm of the form

$$\underset{\sim}{x}_{j+1} = \lambda(\underset{\sim}{i}_s - \underset{\sim}{H}(\underset{\sim}{x}_j)) + \underset{\sim}{x}_j \triangleq \underset{\sim}{M}(\underset{\sim}{x}_j) \qquad (11)$$

for some $\lambda>0$. Note that under the assumptions of Theorem 1, if $\underset{\sim}{x}_j \to \underset{\sim}{x} \in L_T$ and $\underset{\sim}{H}$ is continuous, then $\underset{\sim}{i}_s = \underset{\sim}{H}(\underset{\sim}{x})$ and $\underset{\sim}{x}$ globally maximizes $\overline{P}$. By tightening the assumptions a little further, we can guarantee convergence for all

sufficiently small positive λ.

Theorem 2

Strengthen the assumptions of Theorem 1 by supposing further that L_T is closed and $\underset{\sim}{H}$ is uniformly increasing and Lipschitz continuous on L_T. (See (2), (3).) Then for any $\underset{\sim}{i}_s \varepsilon L_T$, any initial guess $\underset{\sim}{x}_0 \varepsilon L_T$, and any $\lambda \varepsilon (0, 2\delta/K^2)$, the sequence generated by (11) converges to $\hat{\underset{\sim}{v}}(\underset{\sim}{i}_s)$.

Remark

Note that Theorem 2 also guarantees existence of a solution to (6) for all $\underset{\sim}{i}_s \varepsilon L_T$, i.e., $\underset{\sim}{H}(L_T) = L_T$.

Proof

Since $L_T \subset L^2_{n,T}$ is closed and $L^2_{n,T}$ is complete, L_T is complete [11]. It remains to show that $\underset{\sim}{M}$ is contractive, i.e., that for some $C < 1$,

$$\| \underset{\sim}{M}(\underset{\sim}{y}) - \underset{\sim}{M}(\underset{\sim}{x}) \|_T \leq C \, \| \underset{\sim}{y} - \underset{\sim}{x} \|_T, \; \forall \underset{\sim}{x}, \underset{\sim}{y} \; \varepsilon \; L_T, \tag{12}$$

to guarantee $\| \underset{\sim}{x}_n - \hat{\underset{\sim}{v}}(\underset{\sim}{i}_s) \|_T \to 0$ by the contraction mapping theorem [3,p.102], [12,p.28]. But

$$\| \underset{\sim}{M}(\underset{\sim}{y}) - \underset{\sim}{M}(\underset{\sim}{x}) \|^2_T =$$

$$\langle \underset{\sim}{y} - \underset{\sim}{x} - \lambda(\underset{\sim}{H}(\underset{\sim}{y}) - \underset{\sim}{H}(\underset{\sim}{x})), \underset{\sim}{y} - \underset{\sim}{x} - \lambda(\underset{\sim}{H}(\underset{\sim}{y}) - \underset{\sim}{H}(\underset{\sim}{x})) \rangle_T =$$

$$\| \underset{\sim}{y} - \underset{\sim}{x} \|^2_T - 2\lambda \langle \underset{\sim}{H}(\underset{\sim}{y}) - \underset{\sim}{H}(\underset{\sim}{x}), \underset{\sim}{y} - \underset{\sim}{x} \rangle_T + \lambda^2 \| \underset{\sim}{H}(\underset{\sim}{y}) - \underset{\sim}{H}(\underset{\sim}{x}) \|^2_T \leq$$

$$(1 - 2\lambda\delta + \lambda^2K^2)\ \|\underset{\sim}{y}-\underset{\sim}{x}\|_T^2 \triangleq C^2(\lambda)\ \|\underset{\sim}{y}-\underset{\sim}{x}\|_T^2\ ,$$

and $$C^2(\lambda) < 1,\ \forall\lambda\varepsilon(0,2\delta/K^2).$$ ■

III. Examples

3.1) Linear Operators and Memoryless Operators

Consider the time-invariant scalar case for simplicity, and let L_T^2 stand for $L_{1,T}^2$.

If F_ℓ is the convolution operator: $v \mapsto a*b$ where $a: \mathbb{R} \to \mathbb{R}$ is absolutely integrable, then for each $T > 0$, F_ℓ is a continuous linear operator: $L_T^2 \to L_T^2$ and therefore Gâteaux (in fact, Fréchet) differentiable. Since F_ℓ is linear $DF_\ell(x) \equiv F_\ell$, and the reader can easily verify that $(DF_\ell(x))^{adj} = F_\ell^{adj}: v(\cdot) \mapsto a(-\cdot)*v(\cdot)$, i.e., the adjoint operation turns the impulse response around in time. Furthermore, $H_\ell: v(\cdot) \mapsto [a(\cdot) + a(-\cdot)]*v(\cdot)$ is strictly increasing on L_T^2 for each $T > 0$ iff $\text{Re}\ \{\hat{a}(j\omega)\} > 0$ for all ω, where $\hat{a}$ is the Fourier transform of a. This follows from a slight modification of [12:pp.25, 174,235]. Similar results hold if $a(\cdot)$ contains impulse functions as well [12:pp.246-247]. Thus $G_{opt}: v(\cdot) \mapsto a(-\cdot)*v(\cdot)$, and G_{opt} is represented in the frequency domain by the complex admittance $\hat{a}^*(j\omega)$. Therefore Theorem 1 and equation (10) reduce to the standard result $Y_{load}(j\omega) = Y^*_{source}(j\omega)$ if F_ℓ is linear and time-invariant.

Suppose F_m is memoryless but possibly nonlinear, i.e., N_1 is a resistor with the constitutive relation $i=b(v)$. Assume that $b: \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative $b'(\cdot)$ is bounded. Then b is Lipschitz continuous on $\mathbb{R}$ and hence for each $T > 0$ the operator $F_m: v(t) \mapsto b(v(t))$ maps L_T^2 into L_T^2. Using Prop. 13 of [13:p.85] and the Lebesgue Convergence Theorem [13:p.88],

one can show that F_m is Gâteaux differentiable on L_T^2 and that for all $x,y \in L_T^2$, $(DF_m(x))y = y(\cdot)b'(x(\cdot)) \in L_T^2$. Furthermore if $h: v \mapsto b(v)+vb'(v)$ is a strictly increasing function on $\mathbb{R}$, then $H_m: v(t) \mapsto b(v(t)) + v(t)b'(v(t))$ is a strictly increasing operator: $L_T^2 \to L_T^2$. Thus Theorem 1 reduces in this case to the result in [2].

The reader can easily check that $DF_m: L_T^2 \to \mathcal{L}(L_T^2, L_T^2)$ is <u>not</u> continuous unless $b'(\cdot)$ is constant. Thus if N_1 is a resistor with <u>any</u> nonlinearity (other than the trivial $i=gv+\tilde{i}$), F_m is not Fréchet differentiable [4,Chap.3] on L_T^2. This is the reason Theorem 1 was formulated in terms of the weaker Gâteaux derivative.

3.2) Positive Linear Combinations of Operators

The (noncausal) matched load (10) for the source admittance $\underset{\sim}{F}$ is related to $\underset{\sim}{F}$ by a mapping ℓ, $\ell(\underset{\sim}{F}) = \underset{\sim}{G}_{opt}: \underset{\sim}{v} \mapsto (\underset{\sim}{D}\underset{\sim}{F}(\underset{\sim}{v}))^{adj}\underset{\sim}{v}$. Note that ℓ is linear; i.e. $\ell(a\underset{\sim}{F}_1 + b\underset{\sim}{F}_2) = a\,\ell(\underset{\sim}{F}_1) + b\ell(\underset{\sim}{F}_2)$. Given $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$: $L_T \to L_T$, consider $\underset{\sim}{F} \triangleq a\underset{\sim}{F}_1 + b\underset{\sim}{F}_2$. The reader can easily verify that if $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ satisfy the conditions of Theorem 1 (resp. Theorem 2), then $\underset{\sim}{F}$ also satisfies Theorem 1 (resp. Theorem 2), provided $a \geq 0$, $b \geq 0$, $a+b > 0$.

For example, consider the source shown in Fig. 2, where N_1 consists of the parallel connection of an LTI 1-port and a nonlinear resistor. If Y and g satisfy the conditions in section 3.1), then the (noncausal) matched load has the form shown in Fig. 2.

3.3) Circuit Example

Suppose the source takes the specific form in Fig. 3, with the resistor curves shown in Fig. 4. The convolution kernel $a(t) = e^{-t}$, $t \geq 0$, for the

series connection of inductor and resistor satisfies the assumptions of section 3.1. The resistor curves g_K are differentiable everywhere and

$$h_k(v) \triangleq g_k(v) + vg_k'(v) = (k+1)v|v|^{k-1}, \quad k = 1,2,3, \tag{13}$$

with $h_1(0) = 0$. All the assumptions of section 3.1 are satisfied except that the derivatives $g_2'(\cdot)$ and $g_3'(\cdot)$ are unbounded. (Since they are bounded on every bounded subset of $\mathbb{R}$, a more detailed argument, omitted here, shows that the solutions obtained below maximize $\overline{P}$ over $L_T^\infty \cap L_T^2$, which is certainly sufficient in practice.)

To find the optimal output $\hat{v}$ in the three cases, we carried out the iterative procedure (11), which becomes in this instance

$$x_{j+1}(t) =$$

$$\lambda \left[6\sin(t) - (k+1)x_j(t)|x_j(t)|^{k-1} - \int_{-\infty}^{\infty} e^{-|t-\tau|} x_j(\tau)d\tau \right] + x_j(t), \quad k = 1,2,3. \tag{14}$$

Miss Pearl Yew of MIT has written a program in PASCAL to do the numerical solution. It was run on the DEC 20 in MIT's Research Laboratory of Electronics with an initial guess of $x_0(\cdot) \equiv 0$, and found to converge fairly rapidly for small positive values of λ. The results are shown in Fig. 5.

Since g_1 represents a linear resistor, it follows from the traditional linear theorem that $\hat{v}(t) = 2\sin(t)$ for k=1, in agreement with the numerical solution. Note that the instantaneous current drained by the nonlinear source resistor increases in magnitude with k for $|v| > 1$ but decreases for $|v| < 1$. Thus it is intuitively reasonable that the optimal output spends a progressively greater percentage of time in the region $|v| < 1$ as k increases.

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Figure Captions

Fig. 1 The solution of the operator equation (6), given a particular $\underset{\sim}{i_s}(\cdot)$, is the optimal output voltage $\hat{\underset{\sim}{v}}(\cdot)$. It can be achieved with a variety of loads.

Fig. 2 The optimal load admittance is obtained by a linear operator ℓ on the source admittance. Thus the optimal load for a parallel connection of source admittances is the parallel connection of the optimal loads for each source separately.

Fig. 3 Theorems 1 and 2 let us numerically determine the optimal output voltage $\hat{v}(\cdot)$ for this circuit when the resistor curves are as shown in Fig. 4.

Fig. 4 The three resistor curves for the circuit in Fig. 3 are $g_k(v) \triangleq v|v|^{k-1}$, $k=1,2,3$, with $g_1(0) \triangleq 0$.

Fig. 5 One period of the optimal output voltages for the circuit in Fig. 3.

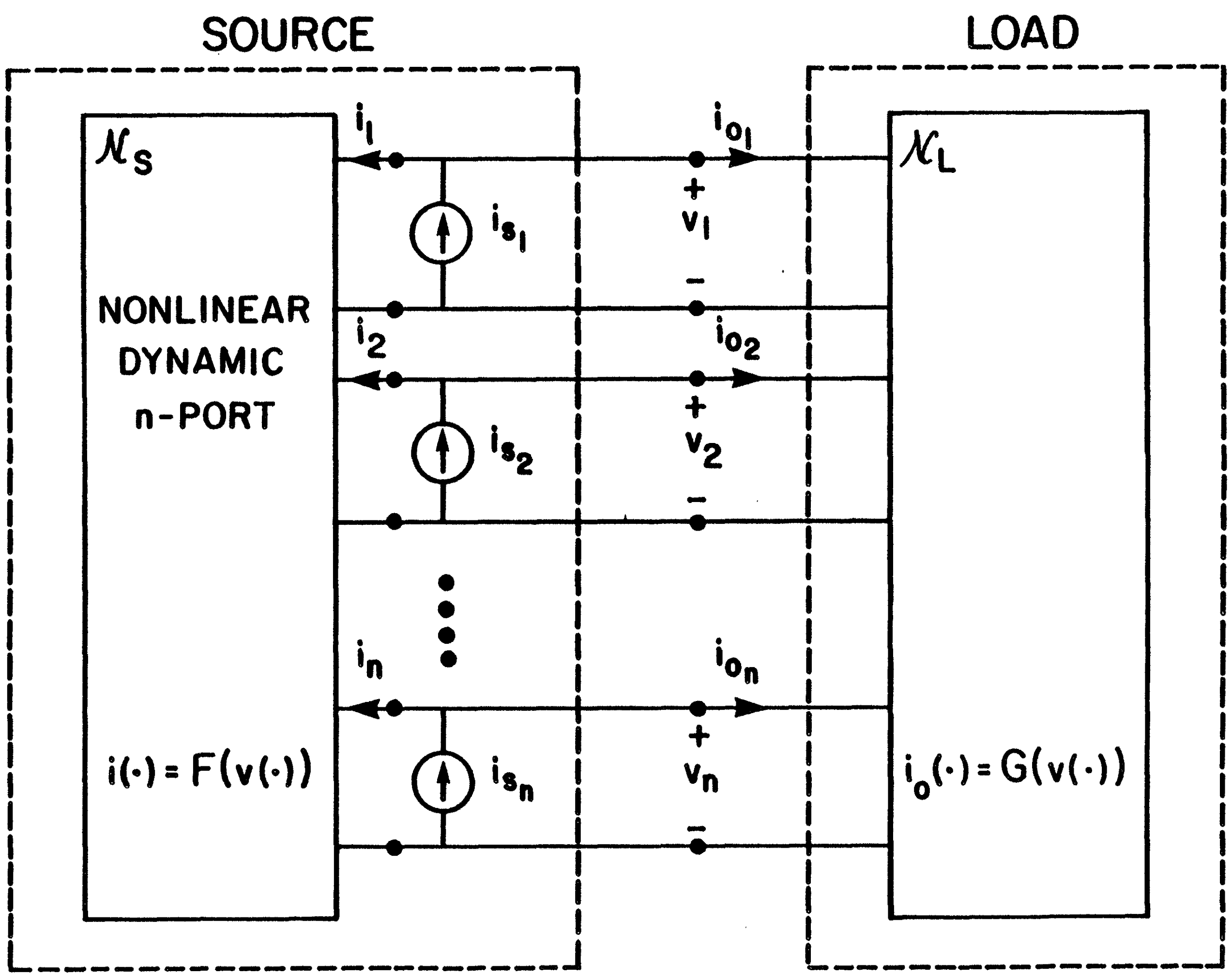
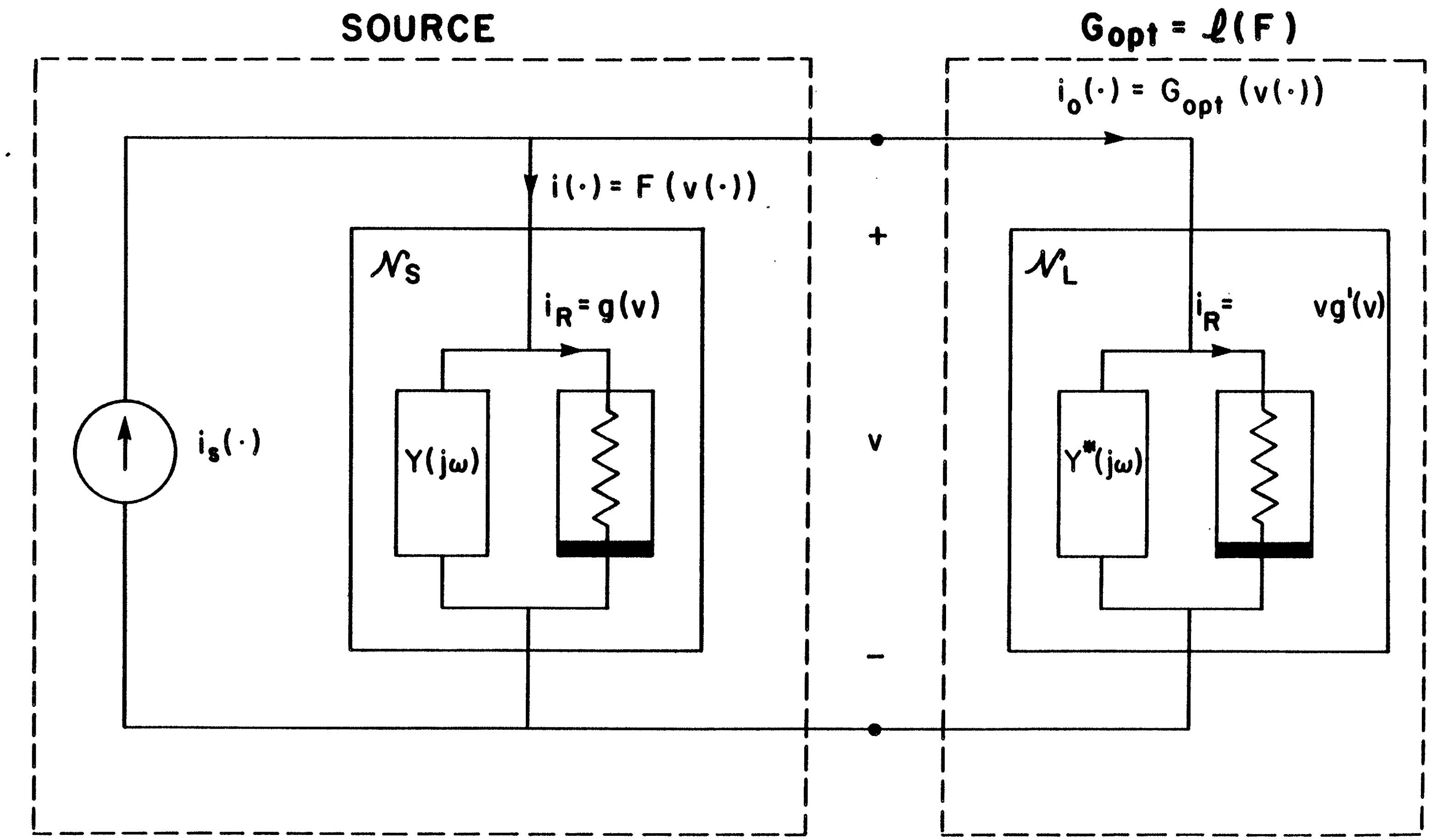

SOURCE
LOAD
$\mathcal{N}_S$
NONLINEAR
DYNAMIC
n-PORT
$i(\cdot) = F(v(\cdot))$
$\mathcal{N}_L$
$i_o(\cdot) = G(v(\cdot))$
i_1
i_2
i_n
i_{s_1}
i_{s_2}
i_{s_n}
i_{o_1}
i_{o_2}
i_{o_n}
+ v_1 −
+ v_2 −
+ v_n −

Figure 1



SOURCE
$G_{opt} = \ell(F)$
$i_o(\cdot) = G_{opt}(v(\cdot))$
$i(\cdot) = F(v(\cdot))$
$\mathcal{N}_S$
$i_R = g(v)$
$i_s(\cdot)$
$Y(j\omega)$
+
v
−
$\mathcal{N}_L$
$i_R =$
$vg'(v)$
$Y^*(j\omega)$

Figure 2

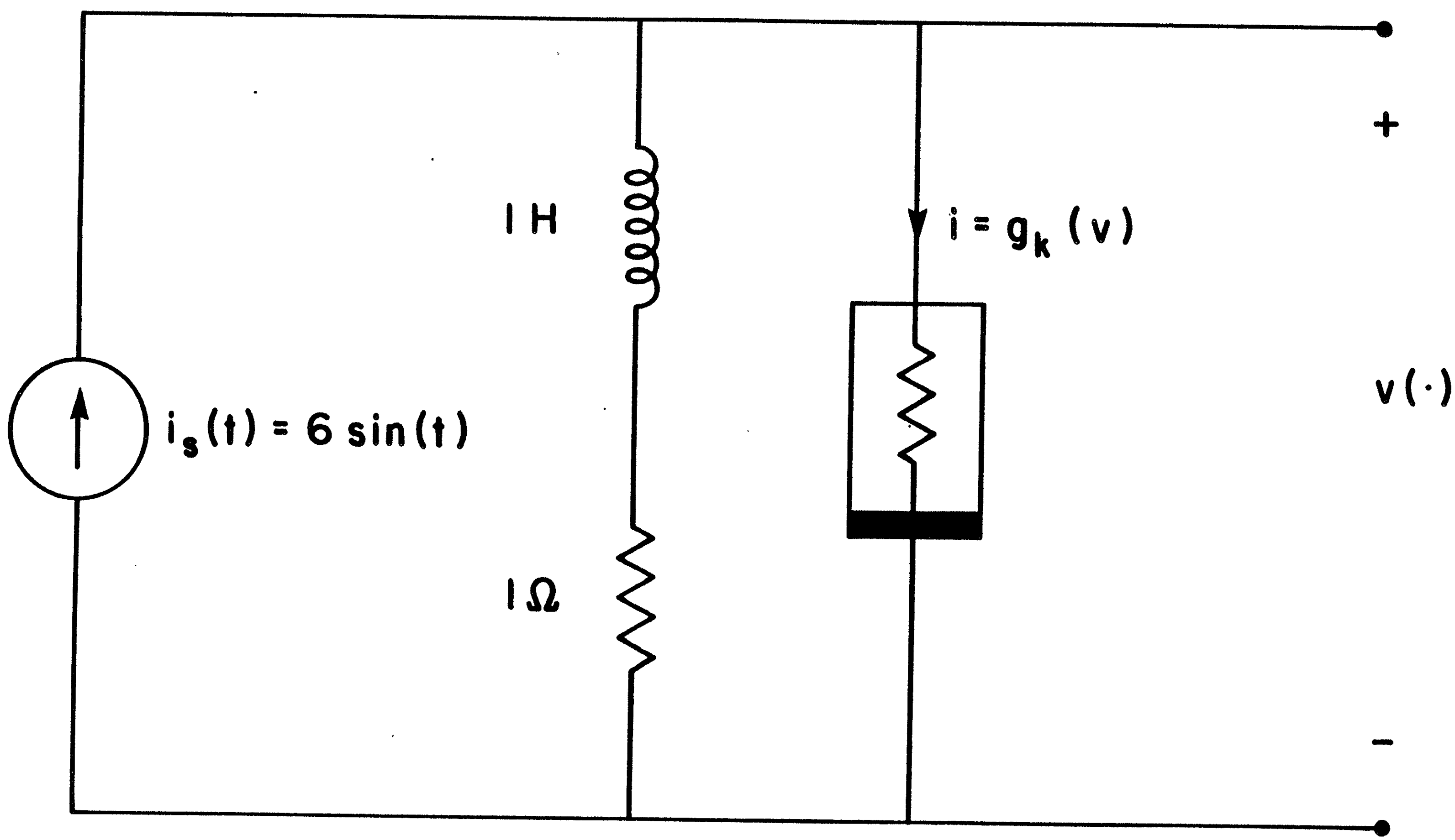


Figure 3

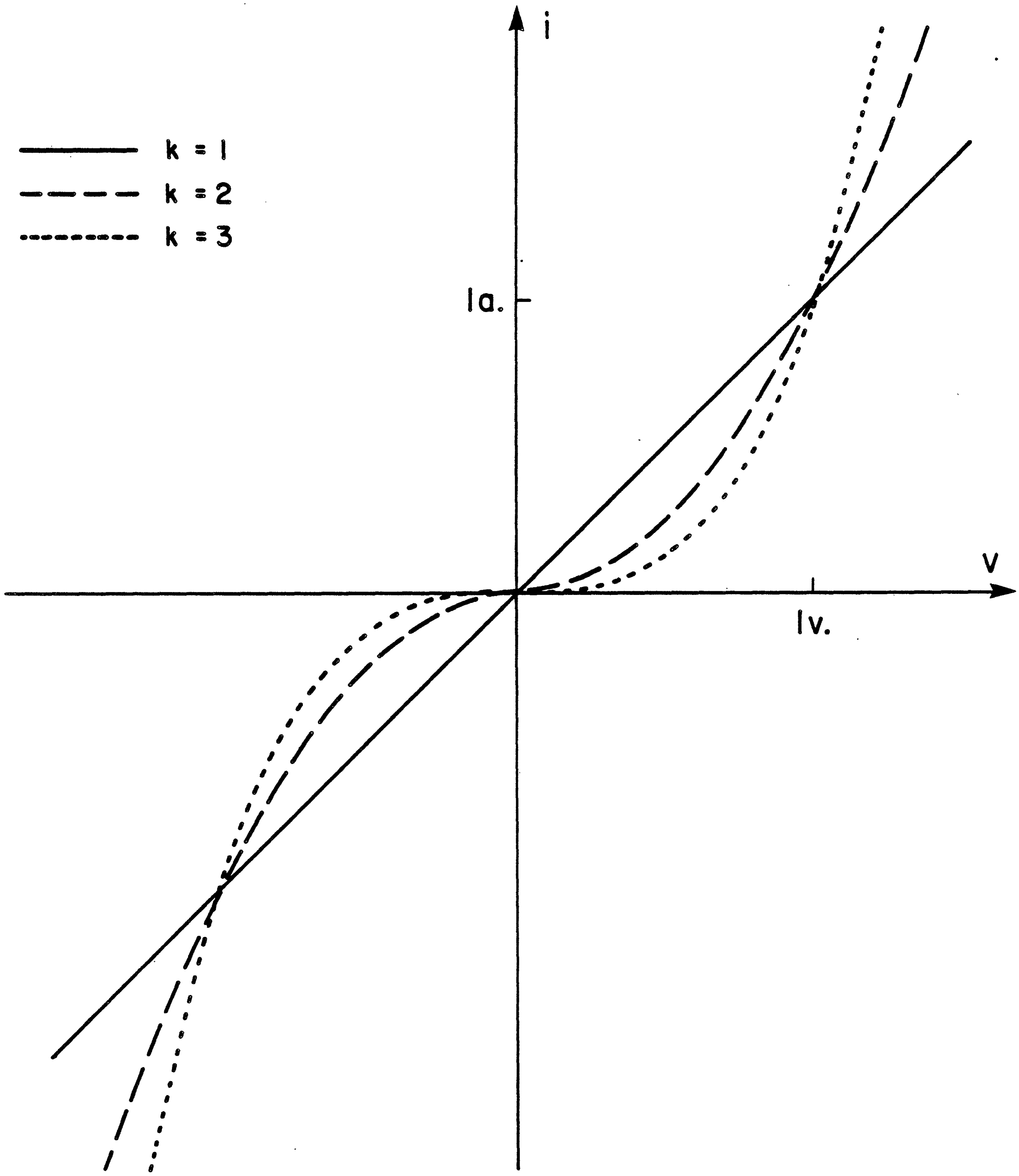


Figure 4

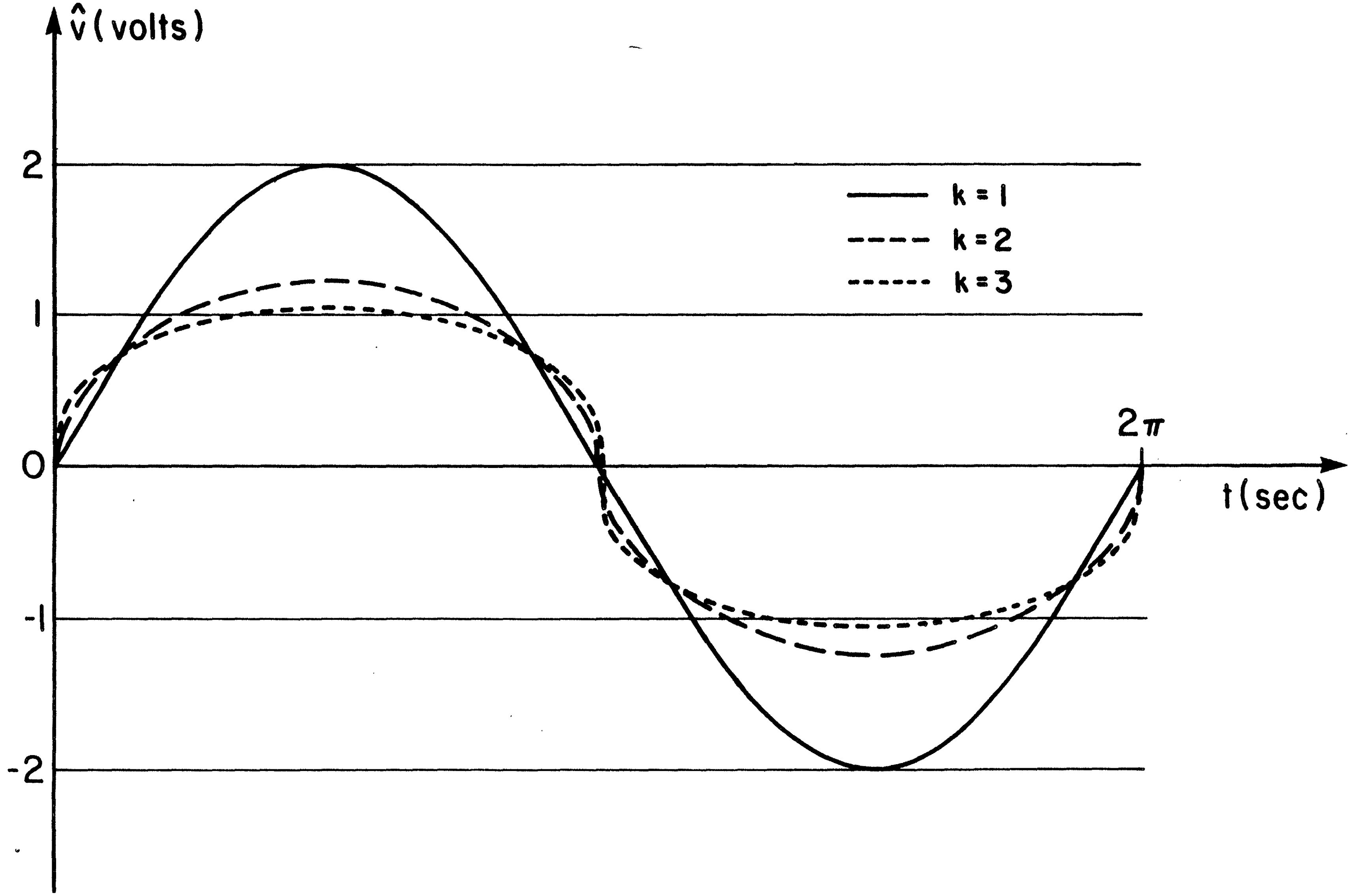


Figure 5